Discrete Mathematics 342 (2019) 562–571

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Partial parking functions

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ARTICLE INFO

Article history:
Received 10 May 2018
Received in revised form 19 October 2018
Accepted 22 October 2018
Available online xxxx

Keywords:
Arrangements of hyperplanes
Shi arrangement
Ish arrangement

ABSTRACT

We characterise the Pak–Stanley labels of the regions of a family of hyperplane arrangements that interpolate between the Shi arrangement and the Ish arrangement.

1. Introduction

In this paper, we characterise the Pak–Stanley labels of the regions of the recently introduced family of the arrangements of hyperplanes "between Shi and Ish" (cf. [6]).

In other words, for $n \in \mathbb{N} = \{1, 2, \dots\}$ there is a labelling (due to Pak and Stanley [13]) of the regions of the n-dimensional Shi arrangement (that is, the connected components of the complement in $\mathbb{R}^n$ of the union of the hyperplanes of the arrangement) by the n-dimensional *parking functions*, and the labelling in this case is a bijection. Remember that the parking functions can be characterised (see Definition 3.3 below; as usual, given $n \in \mathbb{N} \cup \{0\}$, we define $[n] := [1, n]$ where $[m, n] := \{i \in \mathbb{Z} \mid m \leq i \leq n\}$) as

$$\mathbf{a} = (a_1, \dots, a_n) \in [n]^n \text{ such that there is a permutation } \sigma \in \mathfrak{S}_n \text{ with}$$
$$a_{\sigma(i)} \leq i, \text{ for every } i \in [n] .$$

By labelling under the same rules the regions of the n-dimensional *Ish arrangement*, we obtain a new bijection between these regions and the so-called *Ish-parking functions* [4] which can be characterised (see Theorem 3.6 below) as

$$\mathbf{a} = (a_1, \dots, a_n) \in [n]^n \text{ such that there is a permutation } \sigma \in \mathfrak{S}_n \text{ with}$$
$$\begin{cases} a_{\sigma(i)} \leq i, \text{ for every } i \in [a_1] ; \\ \sigma(i+1) < \sigma(i), \text{ for every } i \in [a_1 - 1] . \end{cases}$$

In this paper, we show that the sets of labels corresponding to the arrangements $\mathcal{A}_n^k$ $(2 \leq k \leq n)$ that interpolate between the Shi and the Ish arrangements (which are $\mathcal{A}_n^2$ and $\mathcal{A}_n^n$, respectively) can be characterised (see Proposition 3.10) as

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https://doi.org/10.1016/j.disc.2018.10.032
0012-365X/

$\mathbf{a} = (a_1, \ldots, a_n) \in [n]^n$ such that there is a permutation $\sigma \in \mathfrak{S}_n$ with

$$\begin{cases} a_{\sigma(i)} \le i \text{ for every } i \in [a_1] \text{ and for every } i \in [k, n] \text{ such that } \sigma(i) \ge k\,; \\ \sigma(i+1) < \sigma(i) \text{ for every } i \in [a_1 - 1] \text{ such that } \sigma(i) < k\,. \end{cases}$$

We call these sets of labels *partial parking functions* and note that they all have the same number of elements, viz. $(n+1)^{n-1}$, by [5, Section 2 and Theorem 3.7]. Note that if $k = 2$, $\mathbf{a}$ satisfies the first condition above and $i < a_1$ verifies $\sigma(i) < k$, then $a_1 = a_{\sigma(i)} \le i$, a contradiction.

2. Preliminaries

Consider, for a natural number $n \ge 3$, hyperplanes of $\mathbb{R}^n$ of the following three types. Let, for $1 \le i < j \le n$,

$$\begin{aligned} C_{ij} &= \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j\right\}, \\ S_{ij} &= \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j + 1\right\}, \\ I_{ij} &= \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = x_j + i\right\} \end{aligned}$$

and define, for $2 \le k \le n$,

$$\begin{aligned} \mathcal{A}_n^k := &\left\{C_{ij} \mid 1 \le i < j \le n\right\} \\ &\cup \left\{I_{ij} \mid 1 \le i < j \le n \,\wedge\, i < k\right\} \\ &\cup \left\{S_{ij} \mid k \le i < j \le n\right\} \end{aligned}$$

Note that $\mathcal{A}_n^2 = \mathrm{Shi}_n$, the n-dimensional Shi arrangement, and $\mathcal{A}_n^n = \mathrm{Ish}_n$, the n-dimensional Ish arrangement introduced by Armstrong [1].

2.1. The Pak–Stanley labelling

Let $\mathcal{A} = \mathcal{A}_n^k$ and define, for every (i, j) with $1 \le i < j \le n$,

$$m_{ij} = \begin{cases} 0, & \text{if no hyperplane of equation } x_i - x_j = a \text{ belongs to } \mathcal{A}; \\ \max\{a \mid \mathcal{A} \text{ contains a hyperplane of equation } x_i - x_j = a\}, & \text{otherwise.} \end{cases}$$

Note that

- there are no hyperplanes of equation $x_i - x_j = a$ with $a > 0$ and $i > j$;
- if $a > 0$ and the hyperplane of equation $x_i - x_j = a$ belongs to $\mathcal{A}$, then it also belongs to $\mathcal{A}$ the hyperplane of equation $x_i - x_j = a - 1$.

Similarly to what Pak and Stanley did for the regions of the Shi arrangement (cf. [13]), we may represent a region $\mathcal{R}$ of $\mathcal{A}$ as follows.

Suppose that $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{R}$ and $x_{w_1} > \cdots > x_{w_n}$ for a given $\mathbf{w} = (w_1, \ldots, w_n) \in \mathfrak{S}_n$. Let $\mathcal{H}$ be the set of triples (i, j, a_{ij}) such that $i, j, a_{ij} \in \mathbb{N}$, $1 \le i < j \le n$, $x_i > x_j$, $a_{ij} - 1 < x_i - x_j < a_{ij}$ and the hyperplane of equation $x_i - x_j = a_{ij}$ belongs to $\mathcal{A}$, and let

$$\mathcal{I} = \left\{(i, j) \in \mathbb{N}^2 \mid 1 \le i < j \le n \text{ and } (i, j, a) \notin \mathcal{H} \text{ for every } a \in \mathbb{N}\right\}.$$

Then,

$$\mathcal{R} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x_{w_1} > x_{w_2} > \cdots > x_{w_n}, \\ a_{ij} - 1 < x_i - x_j < a_{ij}\,,\ \forall (i, j, a_{ij}) \in \mathcal{H} \\ x_i - x_j > m_{ij}\,,\ \forall (i, j) \in \mathcal{I} \end{array} \right\}. \tag{2.1}$$

We represent $\mathcal{R}$ by $\mathbf{w}$, decorated with one *labelled arc* for each triple of $\mathcal{H}$, as follows. Given $(i, j, a_{ij}) \in \mathcal{H}$, the arc connects i with j and is labelled a_{ij}, with the following exceptions: if $i \le j < p \le m$, $(i, m, a_{im}), (j, p, a_{jp}) \in \mathcal{H}$ and $a_{jp} = a_{im}$, then we omit the arc connecting j with p. Note that, given $i \le j < p \le m$, forcibly

$$a_{im} > x_i - x_m \ge x_i - x_p \ge x_j - x_p$$

and so $a_{im} \ge a_{jp}$. On the left-hand side of Fig. 1 the regions of Ish_3 are thus represented.

The Pak–Stanley labelling of these regions may be defined as follows. As usual, let $\mathbf{e}_i$ be the i.th element of the standard basis of $\mathbb{R}^n$, $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$.

Definition 2.1. [Pak–Stanley Labelling [13], ad.] Let $\mathcal{R}_0$ be the region defined by

$$x_n + 1 > x_1 > x_2 > \cdots > x_n$$

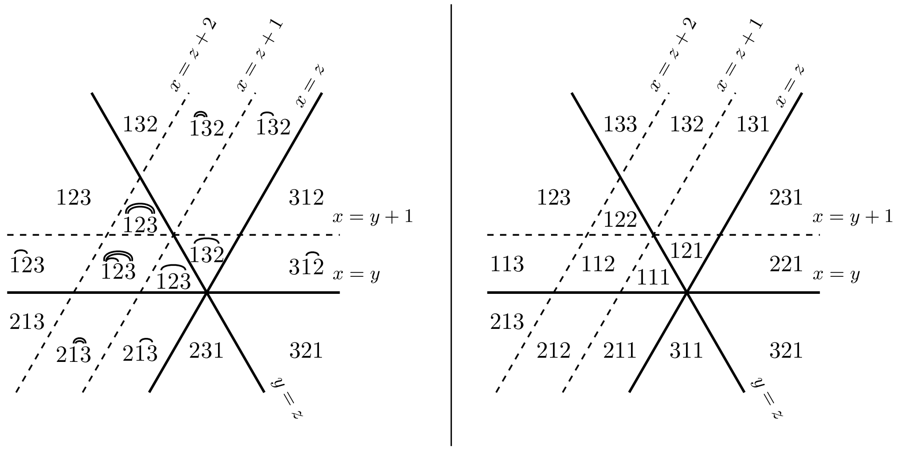


Fig. 1. Pak–Stanley labelling of Ish_3.

(bounded by the hyperplanes of equation $x_j = x_{j+1}$ for $1 \le j < n$ and by the hyperplane of equation $x_1 = x_n + 1$). Then label $\mathcal{R}_0$ with $\ell(\mathcal{A}_n^k, \mathcal{R}_0) := (1, \ldots, 1)$, and, given two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ separated by a unique hyperplane H of $\mathcal{A}_n^k$ such that $\mathcal{R}_0$ and $\mathcal{R}_1$ are on the same side of H, label the regions $\mathcal{R}_1$ and $\mathcal{R}_2$ so that

$$\ell(\mathcal{A}_n^k, \mathcal{R}_2) = \ell(\mathcal{A}_n^k, \mathcal{R}_1) + \begin{cases} \mathbf{e}_i, & \text{if } H = C_{ij} \text{ for some } 1 \le i < j \le n; \\ \mathbf{e}_j, & \text{if } H = S_{ij} \text{ or } H = I_{ij} \text{ for some } 1 \le i < j \le n. \end{cases}$$

Then it is not difficult to directly find the label of a given region (cf. Stanley [13] in the case where $\mathcal{A}$ is the Shi arrangement). Let again $\mathcal{R}$ be defined as in (2.1) and

2.1.1. **take** $\mathbf{t} = \mathbf{t}(\mathbf{w}) = (t_1, \ldots, t_n)$ where $t_{w_i} = \left|\left\{ j \le i \mid w_j \ge w_i \right\}\right|$.
2.1.2. **add** $(a_{ij} - 1)\mathbf{e}_j$ **to t** for every hyperplane $(i, j, a_{ij}) \in \mathcal{H}$.
2.1.3. **add** $m_{ij}\mathbf{e}_j$ **to t** for every pair (i, j) with $1 \le i < j \le n$ and $x_i > x_j$ such that $(i, j, a) \notin \mathcal{H}$ for every $a \in \mathbb{N}$.

In fact, $\mathbf{t}(\mathbf{w})$ is the label of the region of the *Coxeter arrangement*[1] (cf. [12, ad.])

$$\mathcal{R}' = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \middle| x_{w_1} > x_{w_2} > \cdots > x_{w_n} \right\}$$

on the Pak–Stanley labelling, and is also the label of the (unique) region of $\mathcal{A}$ contained in $\mathcal{R}'$ adjacent to the line defined by $x_1 = \cdots = x_n$. Clearly, this region is represented by the permutation $w_1 \cdots w_n$, where all pairs (i, j) such that $1 \le i < j \le n$ and such that there exists in $\mathcal{A}$ a hyperplane of equation $x_i - x_j = a_{ij} > 0$ are covered by a single arc. For example, for every integer $n \ge 2$ and every $2 \le k \le n$, $\ell(\mathcal{A}_n^k, \mathcal{R}_0) = \overparen{12 \cdots n}$.

For every hyperplane that is crossed, either the colour of the arc connecting i and j is increased by one or, if the colour is already as high as possible, the arc disappears. Hence, e.g. the region separated of $\mathcal{R}_0$ by the hyperplane of equation $x_1 = x_n + 1$ is represented by $\overparen{1\,2 \cdots (n{-}1)\;n}$ and its Pak–Stanley label is $11 \cdots 12$.

Note that our representation in the Ish case, since 1 is the initial point of all arcs, is equivalent to the representation already given by Armstrong and Rhoades [2] and used by Leven, Rhoades and Wilson [8].

For another example, let $n = 4$ and consider the region in $\mathcal{A}_4^k$ of label 2311 which is adjacent to the line defined by $x_1 = x_2 = x_3 = x_4$ and contained in

$$\mathcal{R}' = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3 > x_1 > x_4 > x_2 \right\}.$$

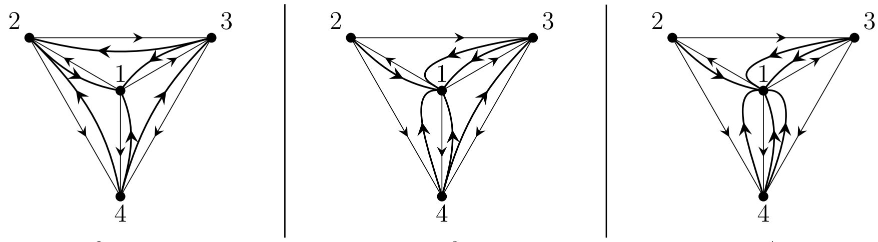
This region is represented by 3142 in $\mathrm{Shi}_4 = \mathcal{A}_4^2$ and in $\mathcal{A}_4^3$, and by 3142 in $\mathrm{Ish}_4 = \mathcal{A}_4^4$. In all the three cases, there are five regions contained in $\mathcal{R}'$ which are described in Table 1.

Note that in all three arrangements the regions labelled 2411 are separated from the region labelled 2311 by the hyperplane of equation $x_1 - x_2 = 1$. The first label is given by (2.1.1) and the second one by (2.1.3). Now, the regions labelled 2411 and 2412 on the left-hand side of the table are separated from each other by the hyperplane of equation $x_3 - x_4 = 1$, whereas the latter is separated from the region labelled 2413 by the hyperplane of equation $x_1 - x_4 = 1$. Hence, 2412 and

[1] I.e., the arrangement $\left\{ C_{ij} \mid 1 \le i < j \le n \right\}$.

Table 1
Labels in $\mathcal{A}_4^k$ of the regions whose points satisfy $x_3 > x_1 > x_4 > x_2$.

$\mathrm{Shi}_4 = \mathcal{A}_4^2$ / $\mathcal{A}_4^3$						$\mathrm{Ish}_4 = \mathcal{A}_4^4$					
Region	$\overparen{3\,1\,4\,2}$	$3\,\overparen{1\,4\,2}$	$\overparen{3\,1\,4}\,2$	$3\,\overparen{1\,4}\,2$	$3\,1\,4\,2$	**Region**	$3\,\overparen{1\,4\,2}$	$3\,\overparen{1\,4}\,2$	$3\,\overparen{\overparen{1\,4}}\,2$	$3\,\overparen{\overparen{\overparen{1\,4}}}\,2$	$3\,1\,4\,2$
Label	2311	2312	2411	2412	2413	**Label**	2311	2411	2412	2413	2414



$\mathcal{G}_4^2 = K_4$ $\mathcal{G}_4^3$ $\mathcal{G}_4^4$

Fig. 2. Directed multi-graphs associated with $\mathrm{Shi}_4 = \mathcal{A}_4^2$, with $\mathcal{A}_4^3$ and with $\mathrm{Ish}_4 = \mathcal{A}_4^4$.

2413 are also labels given by (2.1.3). The regions labelled 2411, 2412, 2413 and 2414 on the right-hand side of the table are separated from one another by the hyperplane of equation $x_1 - x_4 = a$, where $a = 1$ and $a = 2$, and where $a = 3$, respectively. The first two labels, 2412 and 2413, are given by (2.1.2) and the last one, 2414, by (2.1.3).

Finally, note that in Ish_4 the region labelled by 2312 is not contained in $\mathcal{R}'$. In fact, $2312 = 2211 + 0101 = \ell\big(\mathcal{A}_4^4,\ 3\overparen{\overparen{1\,2\,4}}\big)$ since we have $\mathcal{H} = \big\{(1,4,2)\big\}$ for the region $3\overparen{\overparen{1\,2\,4}}$ of $\mathcal{A}_4^4$ and hence $(1,2,a) \notin \mathcal{H}$ for every $a \in \mathbb{N}$ — although in this region $x_1 > x_2$. In both the remaining arrangements, $\mathcal{A}_4^2$ and $\mathcal{A}_4^3$, $\mathcal{H} = \big\{(1,2,1),(1,4,1)\big\}$ for the region $3\,\overparen{1\,4\,2}$, and hence $(3,4,a) \notin \mathcal{H}$ for every $a \in \mathbb{N}$. Yet, the hyperplane of equation $x_3 - x_4 = 1$ belongs to both arrangements.[2]

On the right-hand side of Fig. 1 the Pak–Stanley labelling of the regions of Ish_3 is shown. In dimension n, these labels form the set of *n-dimensional Ish-parking functions*, characterised in a previous article [6]. The labels of the regions of Shi_n form the set of *n-dimensional parking functions*, defined below, as proven by Pak and Stanley in their seminal work [12].

Parking functions and Ish-parking functions, as well as the Pak–Stanley labels of $\mathcal{A}_n^k$ for $2 < k < n$, are *graphical parking functions* as introduced by Postnikov and Shapiro [11] and reformulated by Mazin [9].

3. Graphical parking functions

Definition 3.1 (*[9], ad.*)**.** Let $\mathcal{G} = (V, A)$ be a (finite) directed loopless connected multigraph, where $V = [n]$ for some natural n. Then $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is a *$\mathcal{G}$-parking function* if for every non-empty subset $I \subseteq [n]$ there exists a vertex $i \in I$ such that the number of arcs $(i,j) \in A$ with $j \notin I$, counted with multiplicity, is greater than $a_i - 2$.

Given the arrangement $\mathcal{A}_n^k$, consider a multigraph $\mathcal{G}_n^k$ where for each hyperplane of equation $x_i = x_j$ there is a corresponding arc (i,j), and for each hyperplane of equation $x_i = x_j + a$ with $a \in \mathbb{N}$ there is a corresponding arc (j,i). In Fig. 2, the graphs $\mathcal{G}_4^2$, $\mathcal{G}_4^3$ and $\mathcal{G}_4^4$ are shown. Note that $\mathcal{G}_n^2$ is the complete digraph K_n on n vertices. We will use the following crucial result.

Theorem 3.2 (*Mazin [9], ad.*)**.** *For every $2 \le k \le n$, the set*

$$\big\{\ell(\mathcal{A}_n^k, \mathcal{R}) \mid \mathcal{R} \text{ is a region of } \mathcal{A}_n^k\big\}$$

is the set of $\mathcal{G}_n^k$-parking functions.

3.1. Parking functions

Definition 3.3. The n-tuple $\mathbf{a} = (a_1, \ldots, a_n) \in [n]^n$ is an n-dimensional parking function if[3]

$$\Big|\big\{j \in [n] \mid a_j \le i\big\}\Big| \ge i\,, \quad \forall i \in [n]\,.$$

[2] Note that $\ell\big(\mathcal{A}_4^3,\ 3\overparen{\overparen{1\,2\,4}}\big) = 231\mathbf{3}$.

[3] With this definition, $\mathbf{1} := (1, \ldots, 1) \in [n]^n$ *is* a parking function and $\mathbf{0} := (0, \ldots, 0) \in [0,n]^n$ *is not*. Parking functions are sometimes defined differently, so as to contain $\mathbf{0}$ (and not $\mathbf{1}$). In that case, they are the elements of form $\mathbf{b} = \mathbf{a} - \mathbf{1}$ for $\mathbf{a}$ a parking function in the current sense.

Note that parking functions (sometimes called *classical parking functions*) are indeed $\mathcal{G}_n^2$-parking functions, being $\mathcal{G}_n^2 = K_n$, the complete digraph on $[n]$. In fact, suppose that $\mathbf{a}$ is a K_n-parking function. Then, given $i \in [n]$, let $I = \{j \in [n] \mid a_j > i\}$. If $I = \varnothing$, then $|\{j \in [n] \mid a_j \le i\}| = n \ge i$. If $I \ne \varnothing$, then there is $\ell \in I$ such that $|\{(\ell, j) \in A \mid j \notin I\}| \ge a_\ell - 1$ and so

$$|\{j \in [n] \mid a_j \le i\}| = |\{(\ell, j) \in A \mid j \notin I\}| \ge a_\ell - 1 \ge i\,,$$

the last inequality since $\ell \in I$. The other direction is obvious.

Konheim and Weiss [7] introduced the concept of parking functions that can be thus described. Suppose that n drivers want to park in a one-way street with exactly n places and that $\mathbf{a} \in [n]^n$ is the record of the preferred parking slots, that is, a_i is the preferred parking place of driver $i \in [n]$. They enter the street one by one, driver i immediately after driver $i-1$ parks, directly looks after his/her favourite slot, and if it is occupied he/she tries to park in the first free slot thereafter — or leaves the street if no one exists. Konheim and Weiss showed that $\mathbf{a}$ is a parking function if and only if all the drivers can park in the street in this way.

In other words, consider the following algorithm.

Parking Algorithm

Input: $\mathbf{a} \in [n]^n$

$\texttt{street\_parking} = (0, \dots, 0) \in \mathbb{Z}^{2n}$

foreach $i \in [n]$ in *descending* order **do**

  $p = a_i$

  while $\texttt{street\_parking}(p) \ne 0$ **do**

    increase p

  end while

  $\texttt{parking\_place}(i) = p.$

  $\texttt{street\_parking}(p) = i$

end for

Output: $\texttt{street\_parking}, \texttt{parking\_place}$

We say that $\mathbf{a}$ *parks* $i \in [n]$ if $\texttt{parking\_place}(i) \le n$. Parking functions are those which park every element, or, equivalently, if we set

$$\texttt{first\_free} := \min\{i \in [n+1] \mid \texttt{street\_parking}(i) = 0\} \quad \text{and}$$
$$\texttt{occupied\_positions} = \texttt{street\_parking}^{-1}([n])\,,$$

those for which $\texttt{first\_free} = n + 1$ or those for which $\texttt{occupied\_positions} = [n]$.

Note that by Definition 3.3 $\mathfrak{S}_n$ acts on the set PF_n of n-dimensional parking functions: if $\mathbf{w} \in \mathfrak{S}_n$ and $\mathbf{w}(\mathbf{a}) := \mathbf{a} \circ \mathbf{w} = (a_{w_1}, \dots, a_{w_n})$, then $\mathbf{a} \in \mathsf{PF}_n$ if and only if $\mathbf{w}(\mathbf{a}) \in \mathsf{PF}_n$. In fact, this is a particular case of a more general situation, described in the following result.

Lemma 3.4. *Given* $\mathbf{a} \in [n]^n$ *and* $\mathbf{w} \in \mathfrak{S}_n$,

$$\texttt{occupied\_positions}(\mathbf{a}) = \texttt{occupied\_positions}(\mathbf{a} \circ \mathbf{w})\,.$$

Proof. It is sufficient to prove the claim when $\mathbf{w}$ is the transposition $(i\, i+1)$ for some $i \in [n-1]$. Let $\mathbf{b} := \mathbf{a} \circ \mathbf{w} = (b_1, \dots, b_n) = (a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_n)$, $\alpha := \texttt{parking\_place}(i+1) \ge a_{i+1}$ and $\beta := \texttt{parking\_place}(i) \ge a_i$ when the Parking Algorithm is applied to $\mathbf{a}$.

Suppose that $\beta < \alpha$. Then, since $a_i \le \beta$, $\beta = \texttt{parking\_place}(i)$ and $\alpha = \texttt{parking\_place}(i+1)$ when the algorithm is applied to $\mathbf{b}$. Now, suppose that $\alpha < \beta$. Hence, if $b_{i+1} (= a_i) > \alpha$, then $\beta = \texttt{parking\_place}(i+1)$ and $\alpha = \texttt{parking\_place}(i)$ when the algorithm is applied to $\mathbf{b}$, and if $b_{i+1} \le \alpha$, then $\alpha = \texttt{parking\_place}(i+1)$ and $\beta = \texttt{parking\_place}(i)$. $\square$

3.2. Ish-parking functions

The labels of the regions of Ish_n, the *Ish-parking functions*, are characterised as follows.

Definition 3.5. Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$. The *centre* of $\mathbf{a}$, $Z(\mathbf{a})$, is the (possibly empty) largest set $Z = \{i_1, \dots, i_m\}$ contained in $[n]$ with $n \ge i_1 > \cdots > i_m \ge 1$ and the property[4] that $a_{i_j} \le j$ for every $j \in [m]$.

Theorem 3.6 (*[6, Proposition 3.12]*). *The function* $\mathbf{a} \in [n]^n$ *is an* Ish-parking function *if and only if* $1 \in Z(\mathbf{a})$. $\square$

[4] Note that if this property holds for both $X, Y \subseteq [n]$ then it holds for $X \cup Y$, and so this concept is well-defined (cf. [4–6]). The centre was previously called the *reverse centre* [6].

Proposition 3.7. *Any function* $\mathbf{a} \in [n]^n$ *parks all the elements of* $Z(\mathbf{a})$*. Moreover, for every* $\mathbf{b} \in [n]^n$*, if the restriction to* $Z(\mathbf{a})$ *of* $\mathbf{a}$ *and* $\mathbf{b}$ *are equal, then* $\mathbf{b}$ *also parks all the elements of* $Z(\mathbf{a})$*.*

Proof. Let $Z(\mathbf{a}) = \{i_1, \dots, i_m\}$ with $i_1 > \cdots > i_m$. We show that if $a_{i_j} \le j$ for every $j = 1, \dots, m$ then $\mathbf{a}$ parks all the elements of $Z(\mathbf{a})$. In fact, it is immediate to see by induction on j that when p is assigned a_{i_j} in Line 3 of the Parking Algorithm then $\texttt{street\_parking}(i) \neq 0$ for every $i < p$, and the same happens if we replace $\mathbf{a}$ with $\mathbf{b}$ as described above, since $Z(\mathbf{b}) \supseteq Z(\mathbf{a})$. Hence, $\texttt{first\_free}(\mathbf{a}) > p = a_{i_j}$ and $\mathbf{a}$ parks i_j, and the same holds for $\mathbf{b}$. □

3.3. Partial parking functions

Fixed integers $n \ge 3$ and $1 < k \le n$, and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, consider $\pi \in \mathfrak{S}_n$ such that:

$$\begin{cases} \pi(i) = i, & \text{for every } i < k; \\ a_{\pi(i)} \ge a_{\pi(i+1)}, & \text{for every } k \le i < n; \end{cases}$$

(note that if $k \le i \le n$ then also $k \le \pi(i) \le n$, since $\pi \in \mathfrak{S}_n$). Finally, set

$$\tilde{\mathbf{a}}^k := \mathbf{a} \circ \pi \,.$$

Definition 3.8. $\mathbf{a} \in [n]^n$ is a *k-partial parking function* if:

- $\mathbf{a}$ parks all the elements of $[k, n]$;
- $1 \in Z(\tilde{\mathbf{a}}^k)$.

The restriction to $[k, n]$ of a function $\mathbf{a}$ that parks all the $n + 1 - k$ elements of $[k, n]$ is a particular case of a *defective parking function* introduced by Cameron, Johannsen, Prellberg and Schweitzer [3]. Hence, the number T_k of *all* functions that park every element of $[k, n]$ is $n^{k-1}c(n, n + 1 - k, 0)$, where $c(n, m, k)$ is the number of (n, m, k)-defective parking functions [3, pp. 3], that is

$$T_k = k\, n^{k-1}(n + 1)^{n-k} \,.$$

Lemma 3.9. *A function* $\mathbf{a} \in [n]^n$ *parks every element of* $[k, n]$ *if and only if*

$$\left|\left\{j \in [k, n] \mid a_j \le i\right\}\right| + k - 1 \ge i \,, \quad \forall i \in [k, n] \,.$$

Proof. In fact, since this property does not depend on the first $k - 1$ coordinates of $\mathbf{a}$ we may replace each one of them by 1. Now, the new function parks every element of $[k, n]$ if and only if it is a parking function. □

Proposition 3.10. *A function* $\mathbf{a} = (a_1, \dots, a_n) \in [n]^n$ *is a k-partial parking function if and only if there is a permutation* $\sigma \in \mathfrak{S}_n$ *with*

$$\begin{cases} a_{\sigma(i)} \le i \text{ for every } i \in [a_1] \text{ and for every } i \in [k, n] \text{ such that } \sigma(i) \ge k \,; \\ \sigma(i + 1) < \sigma(i) \text{ for every } i \in [a_1 - 1] \text{ such that } \sigma(i) < k \,. \end{cases}$$

Proof. Let $\tilde{a}_i = a_{\pi(i)}$ be the ith component of $\tilde{\mathbf{a}}^k$ $(1 \le i \le n)$ and $Z = Z(\tilde{\mathbf{a}}^k)$. We suppose that, as in Definition 3.5, $Z = \{\alpha_1, \dots, \alpha_z\}$ with $n \ge \alpha_1 > \cdots > \alpha_z \ge 1$ and $\tilde{a}_{\alpha_i} \le i$ for every $i \in [z]$. Let $B = [k - 1] \setminus Z = \{\beta_1, \dots, \beta_m\}$ and $C = [k, n] \setminus Z = \{\gamma_1, \dots, \gamma_\ell\}$ with $\beta_1 < \cdots < \beta_m$ and $\gamma_1 < \cdots < \gamma_\ell$. Note that, in particular, $a_1 \le z$, $z + m + \ell = n$ and $\tilde{a}_{\gamma_1} \ge \cdots \ge \tilde{a}_{\gamma_\ell}$.

Now, suppose that $\mathbf{a}$ is a k-partial parking function as defined in Definition 3.8. We define $\tau \in \mathfrak{S}_n$ by

$$\tau(t) = \begin{cases} \alpha_t, & \text{if } t \le z; \\ \beta_{t-z}, & \text{if } z < t \le n - \ell; \\ \gamma_{n+1-t}, & \text{if } n - \ell < t \le n. \end{cases}$$

so that $\tau(1) > \cdots > \tau(z)$ and $\tilde{a}_{\tau(n-\ell+1)} \le \cdots \le \tilde{a}_{\tau(n)}$. Finally, we define $\sigma = \pi \circ \tau$.

Then, for every $i \in [a_1] \subseteq [z]$, $a_{\sigma(i)} = \tilde{a}_{\tau(i)} \le i$ and, for every $i \in [a_1 - 1]$ such that $\tau(i) < k$, $\tau(i + 1) < \tau(i) = \sigma(i)$. But then $\tau(i) < k$ implies that $\tau(i + 1) < k$ and thus $\sigma(i + 1) = \tau(i + 1) < \sigma(i)$. Now, suppose that $i < \tilde{a}_{\tau(i)}$ for some $i \in [k, n]$ such that $\tau(i) \ge k$. Then $i \in C$. Since $\tilde{a}_{\tau(i)} \le \tilde{a}_{\tau(j)}$ for every $i < j \le n$ (being, in particular, also $j \in C$),

$$\left|\left\{j \in [k, n] \mid \tilde{a}_j > i\right\}\right| > n - i \,,$$

and thus, contrary to the fact that $\mathbf{a}$ parks all the elements of $[k, n]$ (cf. Lemma 3.9),

$$\left|\left\{j \in [k, n] \mid a_j \le i\right\}\right| + k - 1 < i \,.$$

For example, suppose that $n = 8$, $k = 5$, and $\mathbf{a} = 2663\,1461$. Then $\tilde{\mathbf{a}}^k = 2663\,6411$, $\tau = 87412365$, $\sigma = 85412367$ and $\tilde{\mathbf{a}}^k \circ \tau = \mathbf{a} \circ \sigma = 1132\,6646$.

Conversely, suppose that $a_{\sigma(i)} \le i$ for every $i \in [k, n]$ such that $\sigma(i) \ge k$. By definition of τ, if $i \in [k, n]$ then $i \in Z$ or $\tau(i) > z + m \ge k - 1$. Therefore, $a_{\sigma(i)} = \tilde{a}_{\tau(i)} \le i$ for every $i \in [k, n]$ and hence

$$\left|\left\{\ell \in [k, n] \mid a_\ell \le j\right\}\right| + k - 1 \ge j$$

Finally, $\sigma([a_1]) \cup \{1\} \subseteq Z$ by maximality of Z. $\square$

Indeed, k-partial parking functions are exactly the $\mathcal{G}_n^k$-parking functions. But to prove it we still need a different tool.

4. The DFS-burning algorithm

We want to characterise the $\mathcal{G}_n^k$-parking functions for every $k, n \in \mathbb{N}$ such that $2 \le k \le n$. Similarly to what we did for the characterisation of the Ish-parking functions [6] (the case $k = n$), our main tool is the DFS-Burning Algorithm of Perkinson, Yang and Yu [10] (cf. Fig. 3). Recall that this algorithm, given $\mathbf{a} \in [n]^n$ and a multiple digraph $\mathcal{G}$, determines whether $\mathbf{a}$ is a $\mathcal{G}$-parking function by constructing in the positive case an oriented spanning subtree T of $\mathcal{G}$ that is in bijection with $\mathbf{a}$ [6,10]. The Tree to Parking Function Algorithm (cf. Fig. 3, on the right) builds $\mathbf{a}$ out of T (and $\mathcal{G}$), thus defining the inverse bijection.

Recall [6] that the algorithm is not directly applied to the multidigraph $\mathcal{G}$. Indeed, it is applied to another digraph, $\overline{\mathcal{G}}$, with one more vertex, 0, and set of arcs $\overline{A}$ defined by:

- For every vertex $v \in [n]$, $(0, v) \in \overline{A}$;
- For every arc $(v, w) \in A$, $(w, v) \in \overline{A}$.

We use the following result, which is an extension to directed multigraphs of the work of Perkinson, Yang and Yu [10].

Proposition 4.1 *([6, Proposition 3.2]). Given a directed multigraph $\mathcal{G}$ on $[n]$ and a function $\mathbf{a}\colon [n] \to \mathbb{N}_0$, $\mathbf{a}$ is a $\mathcal{G}$-parking function if and only if the list* `burnt_vertices` *at the end of the execution of the* DFS*-Burning Algorithm applied to $\overline{\mathcal{G}}$ includes all the vertices in $\overline{V} = \{0\} \cup [n]$.*

The different arcs connecting v and w that occur ℓ times ($\ell > 1$) are labelled $(w, v + m\,n) \in \overline{A}$ with $m \in [0, \ell - 1]$, so as to distinguish between them. For this purpose, the DFS-Burning Algorithm inputs the list `neighbours`(w) of vertices v such that $(w, v) \in \overline{\mathcal{G}}$ for each vertex w under the same form, that is, under the form $v + m\,n$ with $m \in [0, \ell - 1]$. However, note that every vertex is seen by the algorithm as a unique entity. In fact, in Line 7 we take $j_n = \mathrm{Mod}(j, n)$ for every $j \in$ `neighbours`(i) (in Line 6).

Note that although the order of the vertices in `neighbours` is not relevant in the context of Proposition 4.1, it is indeed relevant in other contexts, like that of Lemma 4.3 (cf. [6, Remark 3.4.]). We define the order in $\overline{\mathcal{G}_k}$ so that:

1. `neighbours`(0) if formed by the arcs of form $(0, i)$ for every $i \in [n]$; we sort `neighbours`(0) based on the value of i, in descending order.
2. There is an arc of form $(1, i + m\,n)$ for every $i > 1$ and every $0 \le m \le \min\{i, k\} - 2$. We sort `neighbours`$(1)$ by the value of i in descending order, breaking ties by the value of m, again in descending order. For example, in $\mathcal{A}_4^3$ (cf. Fig. 4),

   $$\texttt{neighbours}(1) = \langle 8, 4, 7, 3, 2\rangle\,.$$

3. For every $1 \le m < i$ there is a unique arc (i, m). *Exactly when* $i \ge k$, there is also an arc of form (i, m) for every $i < m \le n$. In all cases, we sort `neighbours`(i) by the value of m in descending order

Example 4.2. We apply the DFS-Burning Algorithm to $\mathbf{a} = 4213 \in [4]^4$ with the three different graphs associated with $n = 4$. Actually, $\mathbf{a}$ is a parking function –that is, a label of a region of $\mathcal{A}_4^2 = \mathrm{Shi}_4$ – since $\widetilde{\mathbf{a}}^2 = 4321$ and $1 \in Z(\widetilde{\mathbf{a}}^2) = [4]$, but neither a label of a region of $\mathcal{A}_4^3$ nor of $\mathcal{A}_4^4 = \mathrm{Ish}_4$, because $\widetilde{\mathbf{a}}^3 = 4231$, $\widetilde{\mathbf{a}}^4 = \mathbf{a} = 4213$, $1 \notin Z(\widetilde{\mathbf{a}}^3) = \{2, 4\}$, and $1 \notin Z(\widetilde{\mathbf{a}}^4) = \{2, 3\}$.

In the first case, where $\texttt{neighbours} = \big\langle\langle 4, 3, 2, 1\rangle, \langle 4, 3, 2\rangle, \langle 4, 3, 1\rangle, \langle 4, 2, 1\rangle, \langle 3, 2, 1\rangle\big\rangle$ (cf. the left table in the bottom of Fig. 4), when the algorithm is applied with $\mathcal{G} = \overline{\mathcal{G}_4^2}$ to $\mathbf{a}$, it calls DFS_FROM(i) with $i = 0$, assigns $j = 4$ and then, since $a_j \neq 1$, $(0, 4)$ is joined to `dampened_edges`. This is represented on the left-hand table below with the inclusion of 0_1 in the top box of column 4. Next assignment, $j = 3$. Since now $a_3 = 1$, $(0, 3)$ is joined to `tree_edges` and DFS_FROM is called with $i = 3$. Then, 0_2 is written in the only box of column 3. At the end, `burnt_vertices` $= \langle 0, 3, 2, 4, 1\rangle$, which proves that 4213 is a $\mathcal{G}_4^2$-parking function, that is, a standard parking function in dimension 4. The respective spanning tree may be defined by the collection of arcs, `tree_edges` $= \langle (0, 3), (0, 2), (2, 4), (0, 1)\rangle$.

We believe that now the content of the tables is self-explanatory. Just note that the entry i_k in column j means that arc (i, j) is the k.th arc to be inserted.[5] Note also that the elements $i \in [n]$ of the bottom row are those for which $a_i = 1$, and thus represent elements from `tree_edges`, whereas the remaining entries represent elements from `dampened_edges`.

5 Perhaps with label $(i, j + m\,n)$.

DFS-BURNING ALGORITHM (AD.)

```
    Input: a: [n] → ℕ
 1: burnt_vertices = {0}
 2: dampened_edges = { }
 3: tree_edges = { }
 4: execute DFS_FROM(0)
    Output: burnt_vertices, tree_edges
    and dampened_edges
AUXILIARY FUNCTION
 5: function DFS_FROM(i)
 6:   foreach j in neighbours(i) do
 7:     j_n = Mod(j, n)
 8:     if j_n ∉ burnt_vertices then
 9:       if a_{j_n} = 1 then
10:         append (i, j) to tree_edges
11:         append j_n to burnt_vertices
12:         execute DFS_FROM(j_n)
13:       else
14:         append (i, j) to dampened_edges
15:         a_{j_n} = a_{j_n} − 1
16:       end if
17:     end if
18:   end for
19: end function
```

TREE TO PARKING FUNCTION ALGORITHM (AD.)

```
    Input: Spanning tree T rooted
 1: at r with edges directed away from root.
 2: burnt_vertices = {r}
 3: dampened_edges = { }
 4: a = (1, ..., 1)
 5: execute TREE_FROM(r)
    Output: a: V \ {r} → ℕ
AUXILIARY FUNCTION
 6: function TREE_FROM(i)
 7:   foreach j in neighbours(i) do
 8:     j_n = Mod(j, n)
 9:     if j_n ∉ burnt_vertices then
10:       if (i, j) is an edge of T then
11:         append j_n to burnt_vertices
12:         execute TREE_FROM(j_n)
13:       else
14:         a_{j_n} = a_{j_n} + 1
15:         append (i, j) to dampened_edges
16:       end if
17:     end if
18:   end for
19: end function
```

Fig. 3. DFS-Burning Algorithm and inverse.

i	0	1	2	3	4
neighbours(i)	4	4	4	4	–
	3	3	3	–	3
	2	2	–	2	2
	1	–	1	1	1

i	0	1	2	3	4
neighbours(i)	4	8 4	–	4	–
	3	7 3	–	–	3
	2	2	–	2	2
	1	–	1	1	1

i	0	1	2	3	4
neighbours(i)	4	12 8 4	–	–	–
	3	7 3	–	–	3
	2	2	–	2	2
	1	–	1	1	1

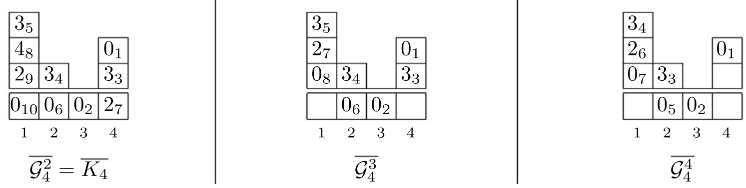


Fig. 4. Lists of neighbours and execution of the DFS-Burning Algorithm.

Finally, note that the algorithm runs in the second graph by choosing the same arcs up to the seventh arc, which is not $(2, 4)$ since $4 \notin$ `neighbours(2)` in this graph. Since `burnt_vertices` $\neq \{0, 1, \ldots, 4\}$ at the end of the execution for the two last graphs, we verify that 4213 is neither a label of the regions of $\mathcal{A}_4^3$ nor an Ish-parking function (in fact, $1 \notin Z(4213) = \{2, 3\}$).

Lemma 4.3. *Let* $\mathbf{a} \in [n]^n$ *be the input of the* DFS-*Burning Algorithm applied to* $\overline{\mathcal{G}_k}$ $(2 \leq k \leq n)$ *as defined above, and suppose that, at the end of the execution, the list of burnt vertices is* `burnt_vertices` $= \langle 0{=}i_0, i_1, \ldots, i_m \rangle$. *Suppose* $i_p := \min\{i_1, \ldots, i_m\} < k$. *Then either* $i_p = 1$ *or* $p = m$. *In any case, if* $\tilde{\mathbf{a}}^k = \mathbf{a} \circ \pi$ *for* $\pi \in \mathfrak{S}_m$ *defined as in the beginning of Section* 3.3

$$Z(\tilde{\mathbf{a}}^k) = \{\pi(i_1), \ldots, \pi(i_p)\}\,.$$

Proof. Note that:

- The value of a_{i_j} is one when i_j is appended to `burnt_vertices`, at Line 11; it has decreased one unit in previous calls of DFS_FROM(i), exactly when $i = i_\ell$ and $i_j \in$ `neighbours`(i_ℓ) for some $\ell < j$. Hence,

$$\forall j \in [m]\,, \quad a_{i_j} \leq j\,.$$

- If $1 < i_j < k$ then $i_{j+1} < i_j < k$, since $i_{j+1} \in$ `neighbours`(i_j) and $i_j < k$.
- If $i_{j+1}, i_j \geq k$, and $i_{j+1} > i_j$, then $a_{i_{j+1}} \leq j$ and $a_{i_j} \leq j + 1$ since $a_{i_j} \leq j$.

Hence, if $\ell_j = \pi(i_j)$ for every $j \in [m]$, then

$$\ell_m \leq \ell_{m-1} \leq \cdots \leq \ell_1\,;$$
$$\forall j \in [m]\,, \quad a_{\ell_j} \leq j\,.$$

For the converse, note that, by definition of $\overline{\mathcal{G}_k}$, if $j \in \texttt{neighbours}(p)$ for some $p > 1$, $m \neq p$, and $m < j$, then also $m \in \texttt{neighbours}(p)$. Thus, if at the end of the execution $j \in \texttt{burnt\_vertices}$, $m < j$ and $a_m \leq a_j + 1$, then also $m \in \texttt{burnt\_vertices}$. $\square$

5. Main theorem

Theorem 5.1. *The $\mathcal{G}_n^k$-parking functions are exactly the k-partial parking functions. Their number is*

$$(n+1)^{n-1}\,.$$

We know that there are $(n+1)^{n-1}$ regions in the $\mathcal{A}_n^k$ arrangement of hyperplanes, which are bijectively labelled by the $\mathcal{G}_n^k$-parking functions [6, Theorem 3.7].

Hence, all we have to prove is the first sentence. This is an immediate consequence of the following Lemma 5.2 and of the fact that the $\mathcal{G}$-parking functions are those functions for which the DFS-Burning Algorithm burns all vertices during the whole execution.

Lemma 5.2. *Let* $\mathbf{a} \in [n]^n$ *be the input of the* DFS-*Burning Algorithm applied to* $\overline{\mathcal{G}_k}$ $(2 \leq k \leq n)$ *as defined above, and consider* $\texttt{burnt\_vertices} = \langle 0{=}i_0, i_1, \ldots, i_m\rangle$ *at the end of the execution. Then the following statements are equivalent:*

5.2.1. $\mathbf{a}$ *parks every element of* $[k, n]$ *and* $i_p = 1$ *for some* $1 \leq p \leq m$*;*
5.2.2. $\mathbf{a}$ *is a k-partial parking function;*
5.2.3. as a set, $\texttt{burnt\_vertices} = \{0\} \cup [n]$ *or, equivalently,* $m = n$*.*

Proof.

(5.2.1) $\Longrightarrow$ (5.2.2). Since $\mathbf{a}$ parks all the elements of $[k, n]$, it is sufficient to show that $1 \in Z(\tilde{\mathbf{a}}^k)$, which follows from Lemma 4.3.

(5.2.2) $\Longrightarrow$ (5.2.3). Suppose that 1 belongs to the centre of $\tilde{\mathbf{a}}^k$ but there is a greatest element $j \in [n]$ which is not in $\texttt{burnt\_vertices}$ at the end of the execution. Suppose first that $j < k$. Then, during the execution of the algorithm (more precisely, during the execution of Line 14) the value of a_j has decreased once for $i = 0$ (that is, as a neighbour of 0), once for each value of $i > j$ (in a total of $n - j$), since $i \in \texttt{burnt\_vertices}$ by definition of j, and $j - 1$ times for $i = 1$, and is still greater than zero. Hence $a_j > n$, which is absurd.

Now, suppose that $j \geq k$, and let

$$\alpha = \min\left\{a_i \mid i \notin \texttt{burnt\_vertices} \cap [k, n]\right\}$$
$$p = \min\left\{q \in [k, n] \mid a_q = \alpha\right\} \quad \text{and}$$
$$A = \left\{q \in [k, n] \mid a_q < \alpha\right\},$$

so that

$$\begin{cases} |A| \geq \alpha - k & (\text{since } \mathbf{a} \text{ parks all the elements of } [k, n]);\\ A \subseteq \texttt{burnt\_vertices}. \end{cases}$$

Again, during the execution of Line 14 the value of $a_p = \alpha$ has decreased once for $i = 0$, once for each value of $i \neq p$ in $\texttt{burnt\_vertices} \cap [k, n] \supseteq A$, and $k - 1$ times for $i = 1$, and is still greater than zero. This means that $\alpha - 1 - (\alpha - k) - (k - 1) > 0$, which is not possible.

(5.2.3) $\Longrightarrow$ (5.2.1). Contrary to our hypothesis, we admit that all the elements of $[n]$ belong to $\texttt{burnt\_vertices}$ at the end of the execution, but that for some $j \in [k, n]$

$$A_j = \left\{q \in [k, n] \mid a_q > j\right\}$$

verifies

$$|A_j| \geq n - j + 1\,.$$

Remember that $\texttt{burnt\_vertices} = \langle 0{=}i_0, i_1, \ldots, i_n\rangle$ is the ordered list of burnt vertices at the end of the execution and let

$$r = \min\left\{q \in [k, n] \mid i_q \in A_j\right\} \text{ and } p = i_r\,.$$

Then $\alpha = a_p > j$. Since $p \in \texttt{burnt\_vertices}$, the number of elements of form (i, p) of the set $\texttt{dampened\_edges} \cup \texttt{tree\_edges}$ (which is equal to α) must be greater than j. But when p was burned, at most $(n - k + 1) - (n - j + 1) = j - k$ elements of $[k, n]$ different from p were already burned, and even if 0 and 1 were also burned, the number of edges could not be greater than $(j - k) + 1 + (k - 1) = j$, a contradiction. $\square$

Acknowledgements

This work was partially supported by CMUP (UID/MAT/00144/2019) and CIDMA (UID/MAT/04106/2019), which are funded by FCT (Portugal) with national (ME) and European structural funds through the programs FEDER, under the partnership agreement PT2020.

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